Phase-contrast CT: Fundamental theorem and fast image reconstruction algorithms

Andrei V. Bronnikov

Bronnikov Algorithms, PO Box 77, Arnhem 6800 AB, The Netherlands

ABSTRACT

Phase-contrast x-ray computed tomography (CT) is an emerging imaging technique that can be implemented at third generation synchrotron radiation sources or by using a microfocus x-ray tube. Promising experimental results have recently been obtained in material science and biological applications. At the same time, the lack of a mathematical theory comparable to that of conventional absorption-based CT limits the progress in this field. We suggest such a theory and prove a fundamental theorem that plays the same role for phase-contrast CT as the Fourier slice theorem does for absorption-based CT. The fundamental theorem allows us to derive fast image reconstruction algorithms in the form of filtered backprojection (FBP).

Keywords: phase contrast, micro CT, tomographic image reconstruction

1. INTRODUCTION

Conventional x-ray computed tomography (CT) is based on the difference in radiation absorption by different tissues. At the same time, a wide range of samples used in biology and medicine demonstrate very weak absorption contrast, nevertheless producing significant phase shifts in the x-ray beam. The use of phase information for imaging purposes is therefore a suitable alternative here. Utilizing phase contrast also has attractive sides itself: first, refractive properties of the medium can be studied, rather than its absorption properties, as done in absorption-based CT and secondly, it may help to diminish the total absorbed dose, enhancing the conditions of the entire imaging procedure.

In this paper we present a mathematical theory which lays down the foundations of quantitative phase-contrast CT, making accurate reconstruction of phase-contrast data as easy as in conventional CT. The suggested theory requires no intermediate step of phase retrieval and provides direct reconstruction of the refractive index from the intensity distributions measured in a single plane of the near field region. In the case of a mixed phase and amplitude object, the data in the contact print plane are required as well. The theory is based on a fundamental relation between the three-dimensional (3D) Radon transform of the object function and the two-dimensional (2D) Radon transform of the phase-contrast projection that is established in the form a fundamental theorem. Using this theorem, reconstruction algorithms can be derived in the form of filtered backprojection.

2. PHASE-CONTRAST IMAGING

Phase-contrast images can be obtained by implementing the Gabor's principle of in-line holography [1]. Pure absorption contrast, which is used in conventional CT, is observed only in the contact print plane (i.e. when the distance z of the object to the image plane is zero), whereas phase contrast due to diffraction of x-rays occurs throughout the entire Fresnel diffraction region (see Fig. 1). To acquire a phase-contrast image it is sufficient to place a detector away from the object. An experimental setup can be built on a synchrotron radiation source or on a microfocus x-ray tube [2-5].

A mathematical model of inline phase-contrast imaging is given by the Fresnel propagator. The intensity distribution at distance z from the object can be represented by

$$I_\theta^z(x,y) = |h_z * U_\theta|^2, \tag{1}$$

The author's web page: www.bronnikov-algorithms.com

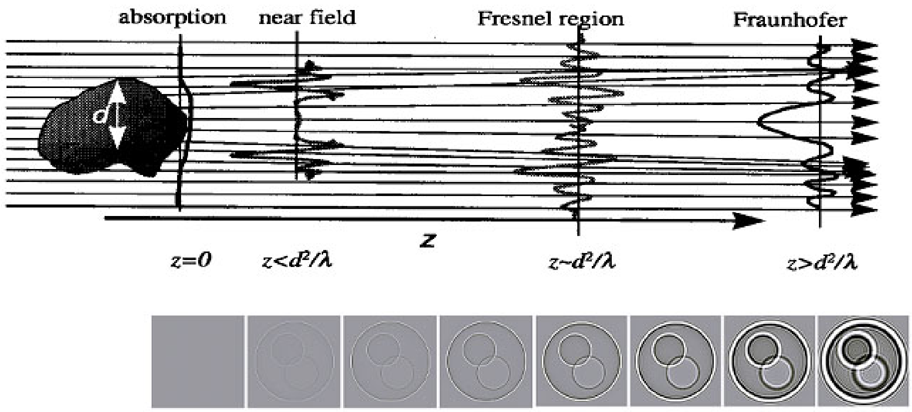

Figure 1. The principle of inline phase-contrast imaging. The projection images of the computer phantom of the spheres are shown for different positions of the detector along the z axis.

where $h_z(x, y)$ is the Fresnel propagator, the asterisk denotes two-dimensional convolution and U_θ is the wavefield downstream of the object at the observation angle θ. Supposing that the detector is in the near field of the Fresnel region:

$$\lambda d << D^2 \tag{2}$$

(where λ is the wavelength, d is the distance of the detector to the object and D is the size of the object) and absorption μ is weak and slowly varying, for the small distance d we may write :

$$I_\theta^d(x, y) = I_\theta^0(x, y)\left(1 - \frac{\lambda d}{2\pi}\nabla^2\varphi_\theta(x, y)\right), \tag{3}$$

where $I_\theta^0(x, y)$ is the intensity in the contact print plane and $\varphi_\theta(x, y)$ is the phase function. Eq. (3) establishes a linear relation between the phase function and the measured intensity data. If Eq. (3) holds, then the fundamental theorem relating the object function to the intensity measured at distance d can be established.

3. CONVENTIONAL CT AND FOURIER SLICE THEOREM

The problem of conventional absorption-based CT is to reconstruct a 3D distribution of the attenuation coefficient μ from the projection data. Mathematical theory of conventional CT is based on the so-called "Fourier slice theorem". Let $\hat{g}_\theta(\xi, \eta)$ be the Fourier transform of the projection at the angle θ. Then

$$\hat{\mu}_\theta(\xi, \eta) = \hat{g}_\theta(\xi, \eta) \tag{4}$$

is the Fourier transform of the attenuation coefficient in the plane that intersects the origin and is parallel to the detector plane. This simple result allows one to find the Fourier transform of the object function by covering the complete Fourier space with the planes positioned at the angles $0 \leq \theta < \pi$. However, the implementation of such an algorithm requires transformation from the polar to Cartesian coordinate system and therefore is not straightforward. It can be shown that after some calculations, the Fourier slice theorem gives a simple algorithmic result in the form of a convolution (denoted by the asterisk) and the backprojection operator:

$$\mu(x_1, x_2, x_3) = \int_0^\pi d\theta\ r * g_\theta, \tag{5}$$

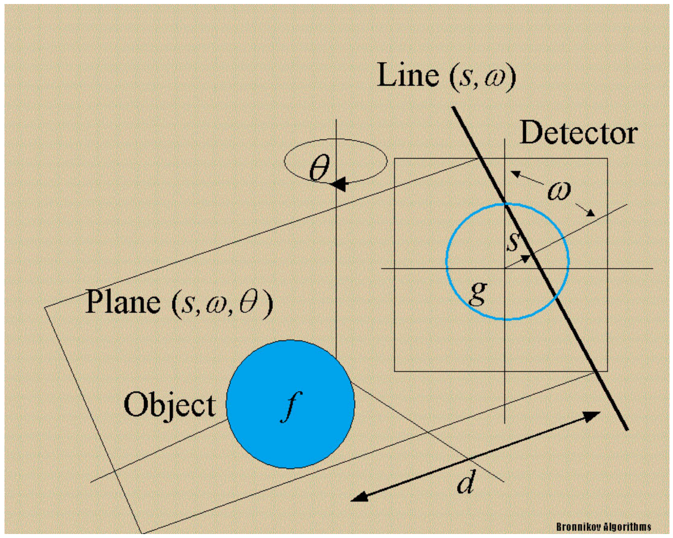

Figure 2. Illustration of the geometry used in the fundamental theorem. The 2D Radon transform of the phase-contrast projection is computed by integration along the lines on the detector; the 3D Radon transform of the object function is computed by integration over the corresponding planes.

where r is the reconstruction filter and following the convolution arguments x and y of function $r * g_\theta$ are replaced by $x = x_1 \cos\theta + x_2 \sin\theta$ and $y = x_3$, respectively. The filter function r has a simple structure $R(\xi) = |\xi|$ in the Fourier domain and is often called "the ramp filter". The ramp filter needs some regularization at the high frequencies where it can amplify the unwanted high-frequency noise.

4. INVERSE PROBLEM IN PHASE-CONTRAST CT

In phase-contrast CT, the data function is computed as

$$g_\theta(x,y) = I_\theta^d(x,y)/I_\theta^0(x,y) - 1, \tag{6}$$

where $I_\theta^d(x,y)$ is the intensity distribution at a sufficiently small distance $z = d$ and $I_\theta^0(x,y)$ is the intensity in the contact print plane. Using $g_\theta(x,y)$ as the data for Eq. (5) we can reconstruct an approximation to the phase object function that is known to be a function proportional to the Laplacian of the object function. Indeed, this function will represent only the edges of the true image, giving no quantitative information. To reconstruct the object function quantitatively, a suitable mathematical model has to be applied. In this way, a formula similar to Eq. (5) has to be found using the apparatus of the Fresnel transform.

The reconstruction problem in quantitative phase-contrast CT is to find the object function $f(x_1, x_2, x_3)$ from measured values of intensity $I_\theta^z(x,y)$, $0 \le \theta < \pi$. A number of methods for solving the inverse problem of phase-contrast CT has been suggested in the literature. These methods can be divided into two groups: a) the methods that require phase retrieval; b) direct methods. A typical example of the phase retrieval method is the holotomography method suggested by Cloetens et al [5]. Several planes of intensity measurements are used here to find the phase distribution. After that the object function is computed by inverting the Radon transform. A direct method that requires no phase retrieval was suggested by the author in [6]. Here the object function (the distribution of the index of refraction of the object) is found directly from the intensity data. Modifications of this method were developed later in [7-11]. For reconstruction of the phase object the direct method requires a measurement of intensity in a single plane of the near field. The data are processed in a way similar to that of conventional CT. The theoretical background of this approach is given by the theorem presented below.

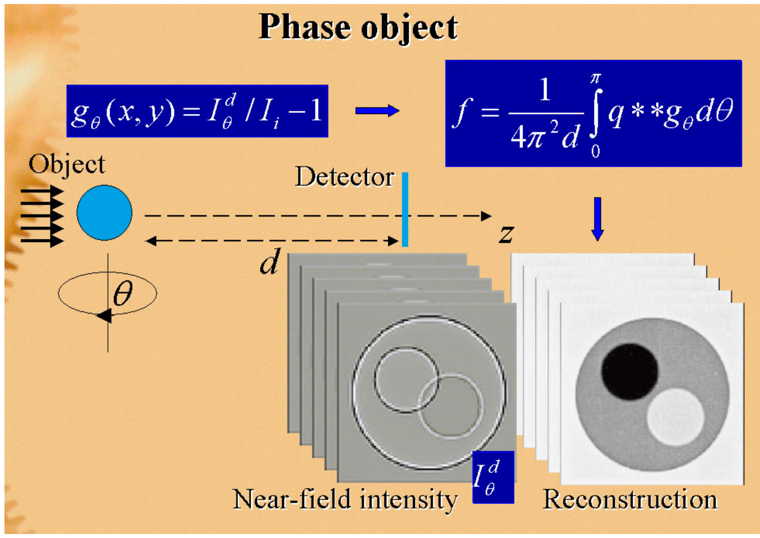

Figure 3. Reconstruction of the phase object by the suggested algorithm. No phase retrieval is required; a single detector position in the near field of the Fresnel region is sufficient.

5. FUNDAMENTAL THEOREM OF PHASE-CONTRAST CT

In order to find an analytical solution to the inverse problem of phase-contrast CT, we make certain approximations that hold in the near field of the Fresnel region (see [7] for details). Using the hat for notations of 2D and 3D Radon transforms, we present the main result in the form of the following

Theorem

Let the data function be given by Eq. (6) and the conditions of Eqs (2) and (3) hold, then

$$\frac{\partial^2}{\partial s^2}\hat{f}(s,\theta,\omega) = \frac{-1}{d}\,\hat{g}_\theta(s,\omega). \tag{7}$$

The theorem shows the relationship between the measured intensity of the x-ray beam downstream of the object and the index of refraction of the object. The theorem has a structure reminding that of the Fourier slice theorem in conventional CT, but instead of the Fourier transform the Radon transform is applied here. The Radon transform is an integration along lines in 2D and an integration over planes in 3D. As seen from Fig. 2, the theorem says that the integrals of the data along lines in the detector plane are proportional to the second derivative of the corresponding plane integrals of the object. Since the formula for the inverse 3D Radon transform is known, the theorem can be directly used to find the object function.

6. RECONSTRUCTION ALGORITHMS

Using the fundamental theorem, a reconstruction algorithm can be derived. Indeed, the object function can be found from equation (7) if the 3D Radon transform is inverted. This gives us the formula

$$f(x_1,x_2,x_3) = \frac{1}{4\pi^2 d}\int_0^\pi \sin\omega d\omega \int_0^\pi d\theta\,\hat{g}_\theta(s',\omega). \tag{8}$$

Inversion formula (8) suggests that a 2D Radon transform in the detector plane has to be computed for each projection $g_\theta(x,y)$. The result is then back-projected into the 3D space by using the backprojection operator of the 3D inverse Radon transform. Applying the full 2D Radon transform to each projection is a lengthy computational task; a more practical FBP reconstruction algorithm could be obtained if Eq. (8) was simplified

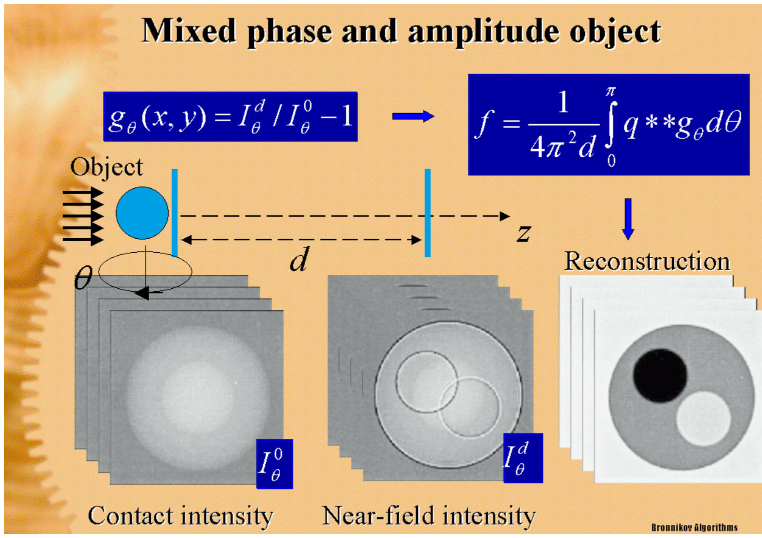

Figure 4. Reconstruction of the mixed phase and amplitude object by the suggested algorithm. Two detector positions: in the contact print plane and in the near field of the Fresnel region are required.

by calculating the integral over angle ω. This result was first obtained in [6]. The algorithm is reduced to a familiar FBP algorithm with the filter function $q(x, y) = |y|/(x^2 + y^2)$:

$$f(x_1, x_2, x_3) = \frac{1}{4\pi^2 d} \int_0^\pi d\theta \; q ** g_\theta, \tag{9}$$

where following the convolution arguments x and y of function $q ** g_\theta$ are replaced by $x = x_1 \cos\theta + x_2 \sin\theta$ and $y = x_3$, respectively. Eq. (9) gives us an algorithm that is similar to the FBP algorithm of conventional CT (see Eq. (5)). The major difference is that the filtering operation in phase-contrast CT is done in two dimensions. The filter can be implemented in the Fourier domain:

$$Q(\xi, \eta) = \frac{|\xi|}{\xi^2 + \eta^2}, \tag{10}$$

where ξ and η are the spatial frequencies.

7. MIXED PHASE AND AMPLITUDE OBJECTS

The use of the approach for mixed phase and amplitude objects is straightforward, at least theoretically. Indeed, since $I_\theta^0(x, y)$ is measured in the contact print plane, it should contain information of absorption contrast only and therefore the absorption-contrast image is canceled by division in Eq. (6). In practical implementation this method may require some additional processing of the data $I_\theta^0(x, y)$ that can be noisy and of insufficient contrast. Note that reconstruction of the mixed phase and amplitude objects will require measurements in two planes, which are the contact print plane and the plane at the distance d from the object. In the case of a purely phase object, $I_\theta^0(x, y) = I_i$, where I_i is the intensity of the incident beam, so that only a measurement in a single plane in the near field is needed (compare Figs. 3 and 4).

8. STABILITY

As was pointed out already in [6,7], $Q(\xi, \eta)$ is a low-pass filter so that the FBP algorithm given by Eqs. (9),(10) is stable to the high-frequency noise. This is a special property of phase-contrast reconstruction. Here we have a situation where the reconstruction algorithm does not need to be stabilized at the high frequencies, which is opposite to the situation in conventional CT. At the same time, the instability of the inverse problem appears

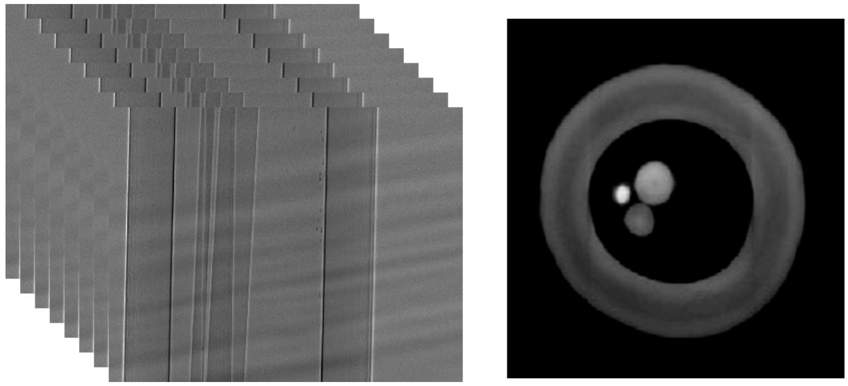

Figure 5. Quantitative phase-contrast reconstruction of the experimental data of the polyethylene tube with polymer fibers. (A data set is shown on the left, a slice from the 3D reconstruction is on the right). The materials with different index of refraction are clearly distinguishable in the reconstructed image.

here in the other way. Note that convolution integral in Eq. (9) is understood in the sense of the Cauchy principal value, since its kernel $q(x, y)$ has a singularity at the origin. This singularity is preserved in the frequency domain, where we also have the singularity at the origin. The singularity problem can be dealt with using a fine sampling in the Fourier domain while setting $Q(\xi, \eta) = 0$ for $\xi = 0, \eta = 0$; in this way the filter allows us to reconstruct the frequencies that are as close to the zero frequencies as possible. A more serious problem is a strong amplification of the low frequencies by the filter function $Q(\xi, \eta)$. The amplification of the low frequencies may lead to the instability if the data contain a low-frequency noise. A simple way to stabilize the behavior of the filter at the low frequencies is to add a regularization parameter to the denominator of $Q(\xi, \eta)$, which gives us

$$Q_\alpha(\xi, \eta) = \frac{|\xi|}{\xi^2 + \eta^2 + \alpha}, \tag{11}$$

where $\alpha > 0$ is a sufficiently small regularization parameter. The use of $Q_\alpha(\xi, \eta)$ makes an approximation to the exact filter, so that the reconstruction is never exact at the low spatial frequencies. This may be unsatisfactory for homogeneous objects, but may work well for the objects with a fine structure. Another problem is the choice of the regularization parameter. According to the author's experience, the FBP algorithm is very sensitive to small changes in parameter α, which requires a special procedure of selecting this parameter for each particular object. Since the stability of the x-ray beams and the presence of absorption remain the main practical issues, the problem of stabilizing the algorithms is one of the most important in implementation of fast FBP algorithms.

As an illustration of the practical applicability of the approach, an experimental data set has been processed. The phase-contrast data were provided by courtesy of A. Groso and M. Stampanoni (Swiss Light Source). Parameters of the data were: d = 15cm, E=13.5 keV, pixel size 1.75 μm. The data have been reconstructed by the suggested algorithm with the regularized filter function. The results are shown in Fig. 5. Here we can see that the algorithm allows us to reconstruct the quantitative image of the object; the materials with different index of refraction are clearly distinguishable.

9. DISCUSSION

A mathematical theory of quantitative phase-contrast computed tomography is presented. A corner stone of this theory is a fundamental theorem that establishes a relation between the 3D Radon transform of the object function and the 2D Radon transform of the phase-contrast projection. The theorem shows a direct relationship between the measured intensity and the index of refraction of the object. The approach requires no phase retrieval and a phase object can be reconstructed using a single detector position in the near field of the Fresnel region. The reconstruction algorithm is derived in the form of filtered backprojection (FBP), which is the fast reconstruction algorithm. The FBP algorithm can be implemented in the Fourier domain using the fast Fourier transform and is as simple as the conventional FBP algorithm. A family of related algorithms can be obtained by modifying the filter function in order to provide the solution with the required properties.

Computer simulations showed promising results both for phase and mixed phase and amplitude objects. A thorough numerical analysis of the reconstruction problem and a series of computer experiments have been done by the author in [7]; we refer to [7] for all necessary numerical illustration of the theory. An independent evaluation of the suggested algorithm was done by Anastasio et al [8]. Groso et al [9] have pointed out the problem of instability of the algorithm to the low frequency noise in the data. Their conjecture was that the real samples are never purely phase objects and therefore some absorption is always present, which can jeopardize the behavior of the reconstruction algorithm at the low spatial frequencies. They have stabilized the reconstruction with respect to the low frequency noise by using a regularized filter function as of Eq. (11) and provided an heuristic procedure for finding the regularization parameter [10]. Gureyev et al suggested a more theoretical approach for selecting the regularization parameter for the case of proportionality of the amplitude and phase objects [11]. A further work is needed to generalize the FBP algorithms for objects with arbitrary absorption.

REFERENCES

1. D. Gabor, "A new microscopic principle," Nature **161** 777-778 (1948).
2. A. Snigirev, I. Snigireva, V. Kohn, S. Kuznetsov, and I. Schelokov, "On the possibilities of x-ray phase contrast microimaging by coherent high-energy synchrotron radiation," Rev. Sci. Instrum. **66** 5486-5492 (1995).
3. T. Davis, D. Gao, T. Gureyev, A. Stevenson, and S. Wilkins, "Phase contrast imaging of weakly absorbing materials using hard x-rays," Nature **373** 595-598 (1995).
4. S. W. Wilkins, T.E. Gureyev, D. Gao, A. Pogany, and A. W. Stevenson, "Phase-contrast imaging using polychromatic hard x-rays," Nature **384** 335-338 (1996).
5. P. Cloetens, W. Ludwig, J. Baruchel, D. Van Dyke, J. Van Landuyt, J. P. Guigay, and M. Schlenker, "Holotomography: quantitative phase tomography with micrometer resolution using hard synchrotron radiation x rays," Appl. Phys. Let. **75** 2912-2914 (1999).
6. A.V. Bronnikov, "Reconstruction formulas in phase-contrast tomography," Optics Communications **171** 239-244 (1999).
7. A.V. Bronnikov, "Theory of quantitative phase-contrast computed tomography," J. Opt. Soc. Am. A **19** 472-480 (2002).
8. M.A. Anastasio, D. Shi, F. De Carlo, and X. Pan, "Analytic image reconstruction in local phase-contrast tomography", Phys. Med. Biol. **49** 121-144 (2004).
9. A.Groso, M. Stampanoni, R. Abela, P. Schneider, S. Linga, R. Muller, "Phase contrast tomography: an alternative approach", Appl. Phys. Let **88** 214104 (2006).
10. A.Groso, R. Abela, M. Stampanoni, "Implementation of a fast method for high resolution phase contrast tomography", Optics Express, to be published (2006).
11. T.E. Gureyev, D.M Paganin, G.R. Myers, Ya.I. Nesterets and S. Wilkins, "Phase-and-amplitude computer tomography," Apll. Phys. Let **89** 034102 (2006).